L_p DERIVATIVES AND APPROXIMATE PEANO DERIVATIVES[1]

BY

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Abstract. It is known that approximate derivatives and kth Peano derivatives share several interesting properties with ordinary derivatives. In this paper the author points out that kth L_p derivatives also share these properties. Furthermore, a definition for a kth approximate Peano derivative is given which generalizes the notions of a kth Peano derivative, a kth L_p derivative, and an approximate derivative. It is then shown that a kth approximate Peano derivative at least shares the property of belonging to Baire class one with these other derivatives.

1. **Introduction.** A real valued function f, defined on an interval, is said to have a kth Peano derivative at x, $k = 1, 2, \ldots$, if there exist numbers $f_1(x), f_2(x), \ldots,$ $f_k(x)$ such that

$$f(x+h) - f(x) - hf_1(x) - \cdots - (h^k/k!)f_k(x) = o(h^k)$$

as $h \to 0$. Calderón and Zygmund [2] have generalized this definition in the following manner. A real valued function f, defined on an interval, is said to have a kth L_p derivative at x, $1 \leq p < \infty$, $k = 0, 1, \ldots$, if there exist numbers $f_{0,p}(x), f_{1,p}(x), \ldots,$ $f_{k,p}(x)$ such that

$$\left\{ \frac{1}{h} \int_0^h \left| f(x+t) - f_{0,p}(x) - tf_{1,p}(x) - \cdots - \frac{t^k}{k!} f_{k,p}(x) \right|^p dt \right\}^{1/p} = o(h^k)$$

as $h \to 0$.

It is known that if a function f possesses a kth Peano derivative everywhere on an interval, then this derivative resembles, in several respects, an ordinary derivative. The following theorem shows that the same remark can be made when a function has a kth L_p derivative everywhere on an interval. Before stating the theorem, let us adopt the notation

$$D^{-1}f(x) = \int_a^x f(t) \, dt,$$

where f is a locally integrable function, and a is any conveniently chosen point. Then employing an argument like the one used in the proof of Theorem 2 in [1], we have

THEOREM 1. *If a measurable function f possesses a kth L_p derivative $f_{k,p}(x)$ for each point x in an interval I, then $f_{k,p}$ is a $(k+1)$th Peano derivative on I. More precisely,*

$$f_{k,p}(x) = (D^{-1}f)_{k+1}(x)$$

for all x in I.

Proof. Using Hölder's inequality we have

$$\left| \frac{1}{h} \int_0^h \left(f(x+t) - f_{0,p}(x) - t f_{1,p}(x) - \cdots - \frac{t^k}{k!} f_{k,p}(x) \right) dt \right|$$

$$\leq \left\{ \frac{1}{h} \int_0^h \left| f(x+t) - f_{0,p}(x) - t f_{1,p}(x) - \cdots - \frac{t^k}{k!} f_{k,p}(x) \right|^p dt \right\}^{1/p} = o(h^k)$$

for each x in I. So we have

$$\int_0^h \left(f(x+t) - f_{0,p}(x) - t f_{1,p}(x) - \cdots - \frac{t^k}{k!} f_{k,p}(x) \right) dt = o(h^{k+1}),$$

and performing the integration, we obtain

$$D^{-1}f(x+h) - D^{-1}f(x) - h f_{0,p}(x) - \cdots - (h^{k+1}/(k+1)!)f_{k,p}(x) = o(h^{k+1}).$$

So $(D^{-1}f)_{k+1}(x) = f_{k,p}(x)$, and the proof is completed.

It is not difficult to see that an ordinary derivative which exists everywhere on an interval is of Baire class one and has the Darboux property. Denjoy [4] and Oliver [11] have shown that a kth Peano derivative which exists on an interval is of Baire class one, and Oliver also showed that a kth Peano derivative has the Darboux property. Denjoy [5] and Clarkson [3] have shown that an ordinary derivative possesses a stronger property than the Darboux property. We shall call this property the Denjoy property, and a concise definition of it is found in [13], where it goes under the name of property A. Oliver [11] has shown that a kth Peano derivative also has the Denjoy property. Zahorski [14] defined a stronger property yet, which we shall call the Zahorski property, and showed that an ordinary derivative possesses it. Weil in [13], where a definition of this property may be found under the name of property B, showed that a kth Peano derivative also has the Zahorski property. Hence we immediately obtain the following result:

COROLLARY 1. *If a measurable function f possesses a kth L_p derivative $f_{k,p}(x)$ at each point x in an interval I, then $f_{k,p}$*
  (1) *is of Baire class one,*
  (2) *has the Darboux property,*
  (3) *has the Denjoy property,*
  (4) *has the Zahorski property.*

2. **Approximate derivatives.**   In order to motivate what follows we recall some of the properties of approximate derivatives.

DEFINITION 1. A function f is said to have an approximate derivative $f'_{\mathrm{ap}}(x)$ at x if

$$\lim_{\substack{\mathrm{ap} \\ h \to 0}} \frac{f(x+h)-f(x)}{h} = f'_{\mathrm{ap}}(x),$$

i.e. if there exists a set E of density 1 at 0 such that

$$\lim_{\substack{h \to 0;\, h \in E}} \frac{f(x+h)-f(x)}{h} = f'_{\mathrm{ap}}(x).$$

Suppose now that f is a function, defined on an interval I, possessing an approximate derivative at each point of I. Tolstoff [12] has shown that under these circumstances f'_{ap} is of Baire class one. Khintchine [7] has shown that Rolle's theorem holds for approximate derivatives. It then readily follows that f'_{ap} has the Darboux property. These proofs are quite long. Shorter ones have been advanced by Goffman and Neugebauer [6]. Marcus [9] has shown that f'_{ap} has the Denjoy property, and Weil [13] has shown that it has the Zahorski property.

3. **Approximate Peano derivatives.**   If we combine the notions of an approximate derivative and a kth Peano derivative, we obtain the following concept.

DEFINITION 2. A function f, defined on an interval, has a kth approximate Peano derivative at a point x, $k = 1, 2, \ldots$, if there exist numbers $f_{(1)}(x), f_{(2)}(x), \ldots, f_{(k)}(x)$ such that

$$\lim_{\substack{\mathrm{ap} \\ h \to 0}} \frac{1}{h^k} \left\{ f(x+h) - f(x) - h f_{(1)}(x) - \cdots - \frac{h^k}{k!} f_{(k)}(x) \right\} = 0.$$

THEOREM 2. *Let f be a measurable function possessing a kth L_p derivative at a point x. Then $f_{k,p}(x)$ is a kth approximate Peano derivative at x; specifically, $f_{k,p}(x) = (f_{0,p})_{(k)}(x)$.*

Proof. We will employ a technique used by Neugebauer in Lemma 7 of [10]. Set

$$g(y) = f_{0,p}(x-y) - f_{0,p}(x) - y f_{1,p}(x) - \cdots - (y^k/k!) f_{k,p}(x).$$

Note that $g(0) = g_{0,p}(0) = g_{1,p}(0) = \cdots = g_{k,p}(0) = 0$. So in order to prove the theorem it will suffice to show that $g_{(k)}(0) = 0$, in particular that

$$\lim_{\substack{\mathrm{ap} \\ h \to 0}} \frac{g(h)}{h^k} = 0.$$

We will show here that $\lim_{\mathrm{ap}\, h \to 0^+} g(h)/h^k = 0$. A similar argument will then show that $\lim_{\mathrm{ap}\, h \to 0^-} g(h)/h^k = 0$.

Let $\varepsilon > 0$ be given, and let $E_\varepsilon = \{ t > 0 : |g(t)| > \varepsilon t^k \}$. We must show that E_ε has 0 as a point of density 0 from the right. For any positive h, let $E_h = E_\varepsilon \cap [0, h]$. Then

we have

$$\left\{\frac{1}{h}\int_0^h |g(t)|^p\, dt\right\}^{1/p} \geqq \left\{\frac{1}{h}\int_{E_h} |g(t)|^p\, dt\right\}^{1/p} \geqq \left\{\frac{1}{h}\int_{E_h} \varepsilon^p t^{kp}\, dt\right\}^{1/p}$$

$$\geqq \left\{\frac{1}{h}\int_0^{|E_h|} \varepsilon^p t^{kp}\, dt\right\}^{1/p} = \left\{\frac{\varepsilon^p}{kp+1}\cdot\frac{|E_h|^{kp+1}}{h}\right\}^{1/p},$$

and as $h \to 0^+$, this is $o(h^k)$, i.e. $\lim_{h\to 0^+} (|E_h|/h)^{k+1/p} = 0$. Hence $\lim_{h\to 0^+} |E_h|/h = 0$, and so

$$\lim_{\substack{\mathrm{ap}\\ h\to 0^+}} \frac{g(h)}{h^k} = 0.$$

With the above theorem we see that a kth approximate Peano derivative is a generalization of all the derivatives mentioned so far in this paper. So we ask whether or not a kth approximate Peano derivative which exists on an interval has any or all of the above-mentioned properties shared by the other derivatives. We will give a partial answer here by showing that such a derivative is at least of Baire class one. We will need a couple of preliminary results.

DEFINITION 7. As in [8] we define differences $\Delta_k(x, h; f)$ for a function f, $k = 1, 2, \ldots$, by

$$\Delta_k(x, h; f) = \sum_{j=0}^k (-1)^{k-j}\binom{k}{j} f(x+jh-\tfrac{1}{2}kh).$$

The following lemma is not difficult to prove using induction on k (see [1] or [8]).

LEMMA 1. *Let λ be any real number. Then*

$$\sum_{j=0}^k (-1)^{k-j}\binom{k}{j}(\lambda+j-\tfrac{1}{2}k)^i = 0, \qquad i = 0, 1, \ldots, k-1,$$

$$= k!, \qquad i = k.$$

LEMMA 2. *If f has a kth approximate Peano derivative at a point x, then for any fixed real number λ there is a set $F(\lambda, x)$ of density 1 at 0 such that*

$$\lim_{h\to 0;\, h\in F(\lambda, x)} \frac{\Delta_k(x+\lambda h, h; f)}{h^k} = f_{(k)}(x).$$

Proof. Let E be a set of density 1 at 0 such that

$$f(x+h) - \sum_{i=0}^k \frac{f_{(i)}(x)}{i!}\, h^i = o(h^k)$$

as $h \to 0$, $h \in E$, where we temporarily denote $f(x)$ by $f_{(0)}(x)$. Let a real number λ be given and define

$$F(\lambda, x) = \{h : \lambda h+jh-\tfrac{1}{2}kh \in E \text{ for each } j = 0, 1, \ldots, k\}.$$

Then $F(\lambda, x)$ is of density 1 at 0. Let $\varepsilon > 0$ be given. There exists a $\delta > 0$ such that if $h \in E$ and $|h| < \delta$, then

$$\left| f(x+h) - \sum_{i=0}^{k} \frac{f_{(i)}(x)}{i!} h^i \right| < \frac{\varepsilon}{k^k} |h|^k,$$

and, furthermore, if I is an interval containing 0, and $|I| < \delta$, then $|E \cap I|/|I| > 1 - \varepsilon$. Consequently, we can find a $0 < \delta' \leq \delta$, such that if $h \in F(\lambda, x)$ and $|h| < \delta'$, then

$$(1) \qquad \left| f(x + \lambda h + jh - \tfrac{1}{2}kh) - \sum_{i=0}^{k} \frac{f_{(i)}(x)}{i!} (\lambda + j - \tfrac{1}{2}k)^i h^i \right| < \frac{\varepsilon}{k^k} |\lambda + j - \tfrac{1}{2}k|^k |h|^k$$

for each $j = 0, 1, \ldots, k$, and if I is an interval containing 0 with $|I| < \delta'$, then $|F(\lambda, x) \cap I|/|I| > 1 - \varepsilon$. If we now consider the right-hand side of inequality (1), we have

$$\frac{\varepsilon}{k^k} |\lambda + j - \tfrac{1}{2}k|^k |h|^k \leq \frac{\varepsilon |h|^k}{k^k} \sum_{n=0}^{k} \binom{k}{n} |\lambda|^n |j - \tfrac{1}{2}k|^{k-n}$$

$$< \frac{\varepsilon |h|^k}{2^k} \sum_{n=0}^{k} \binom{k}{n} |\lambda|^n$$

$$\leq \varepsilon |h|^k |\lambda|^k \quad \text{if } |\lambda| > 1,$$

$$\leq \varepsilon |h|^k \qquad \text{if } |\lambda| \leq 1.$$

Let us suppose that $|\lambda| > 1$ and that $h \in F(\lambda, x)$ with $|h| < \delta'$. Then

$$\left| \frac{\Delta_k(x + \lambda h, h; f)}{h^k} - f_{(k)}(x) \right|$$

$$= \left| \frac{1}{h^k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x + \lambda h + jh - \tfrac{1}{2}kh) - f_{(k)}(x) \right|$$

$$\leq \left| \frac{1}{h^k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left[f(x + \lambda h + jh - \tfrac{1}{2}kh) - \sum_{i=0}^{k} \frac{f_{(i)}(x)}{i!} (\lambda + j - \tfrac{1}{2}k)^i h^i \right] \right|$$

$$+ \left| \frac{1}{h^k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \sum_{i=0}^{k} \frac{f_{(i)}(x)}{i!} (\lambda + j - \tfrac{1}{2}k)^i h^i - f_{(k)}(x) \right|$$

$$\leq \sum_{j=0}^{k} \binom{k}{j} \varepsilon |\lambda|^k + \left| \frac{1}{h^k} \sum_{i=0}^{k} \frac{f_{(i)}(x)}{i!} h^i \sum_{j=0}^{k} (-1)^{k-j} (\lambda + j - \tfrac{1}{2}k)^i - f_{(k)}(x) \right|$$

$$= 2^k |\lambda|^k \varepsilon + |f_{(k)}(x) - f_{(k)}(x)| \quad \text{(by Lemma 1)}$$

$$= 2^k |\lambda|^k \varepsilon.$$

Considering the case where $|\lambda| \leq 1$, we similarly obtain $|\Delta_k(x + \lambda h, h; f)/h^k - f_{(k)}(x)| < 2^k \varepsilon$ if $h \in F(\lambda, x)$ and $|h| < \delta'$. So in either case we have

$$\lim_{h \to 0;\ h \in F(\lambda, x)} \frac{\Delta_k(x + \lambda h, h; f)}{h^k} = f_{(k)}(x).$$

THEOREM 3. *If f is measurable and has a kth approximate Peano derivative at each point of an interval J, then $f_{(k)}$ is of Baire class one.*

Proof. For each positive integer n, each integer p, each real number h, and each real number α, set

$$I_{n,p} = [(p - \tfrac{3}{2})/2^n, (p + \tfrac{3}{2})/2^n], \qquad I_n = [-1/2^{n+1}, 1/2^{n+1}],$$
$$S_{n,p,\alpha,h} = \{x \in I_{n,p} : \Delta_k(x, h; f)/h^k > \alpha\},$$
$$T_{n,p,\alpha} = \{\tfrac{1}{2}kh \in I_n : |S_{n,p,\alpha,h}| > \tfrac{1}{2}|I_{n,p}|\}.$$

For each point of the form $p/2^n \in J$, define

$$f_n(p/2^n) = \sup\{\alpha : |T_{n,p,\alpha}| > \tfrac{1}{2}|I_n|\}.$$

For each fixed n extend f_n linearly to arrive at a continuous function on all of J.

Let $x_0 \in I$. We want to show that $f_n(x_0) \to f_{(k)}(x_0)$ as $n \to \infty$. From Lemma 2 we know there is a set $F(0, x_0)$ of density 1 at 0 such that

$$\lim_{h \to 0;\, h \in F(0, x_0)} \frac{\Delta_k(x_0, h; f)}{h^k} = f_{(k)}(x_0).$$

Set $G = \{\tfrac{1}{2}kh : h \in F(0, x_0)\}$. G is clearly of density 1 at 0.

Let $\varepsilon > 0$ be given. We shall find it convenient later to suppose that $1 - (k+1)\varepsilon > \tfrac{3}{4}$ and $1 - 2\varepsilon^{1/2k} - \varepsilon > \tfrac{1}{2}$. There exists a $\delta > 0$ such that if we let

$$E_\varepsilon = \left\{h : \left| f(x_0 + h) - \sum_{i=0}^{k} \frac{f_{(i)}(x_0)}{i!} h^i \right| < \frac{\varepsilon}{k^k}|h|^k \right\},$$

$$G_\varepsilon = \{\tfrac{1}{2}kh : |\Delta_k(x_0, h; f)/h^k - f_{(k)}(x_0)| < 2^k \varepsilon\},$$

and

$$F_\varepsilon = \{\tfrac{1}{2}kh : jh - \tfrac{1}{2}kh \in E_\varepsilon \text{ for each } j = 0, 1, \ldots, k\},$$

where we have seen in Lemma 2 that $F_\varepsilon \subset G_\varepsilon$, then

$$|F_\varepsilon \cap I|/|I| > 1 - \varepsilon, \quad \text{and} \quad |E_\varepsilon \cap I|/|I| > 1 - \varepsilon$$

for any interval I containing 0 with $|I| < \delta$.

Now choose a positive integer N so large that $1/2^N < \delta/4$. Let $n > N$, and find the unique integer p so that $p/2^n < x_0 \leq (p+1)/2^n$.

Let

$$\tfrac{1}{2}kh \in \left\{ \left[-\frac{1}{2^{n+1}}, -\frac{\varepsilon^{1/2k}}{2^{n+1}} \right] \cup \left[\frac{\varepsilon^{1/2k}}{2^{n+1}}, \frac{1}{2^{n+1}} \right] \right\} \cap F_\varepsilon,$$

and hold it fixed. For each $j = 0, 1, \ldots, k$ we have

$$jh - \tfrac{1}{2}kh \in [-1/2^{n+1}, 1/2^{n+1}] \cap E_\varepsilon.$$

Let $B_j = \{y - jh + \frac{1}{2}kh : y \in E_\varepsilon\}$. Then for each $j = 0, 1, \ldots, k$ we have

$$\frac{|B_j \cap [-3/2^{n+1}, 1/2^{n+1}]|}{1/2^{n-1}} > 1 - \varepsilon.$$

If we set $B = \bigcap_{j=0}^{k} B_j$, then

$$\frac{|B \cap [-3/2^{n+1}, 1/2^{n+1}]|}{1/2^{n-1}} > 1 - (k+1)\varepsilon.$$

Furthermore, if $\lambda h \in B \cap [-3/2^{n+1}, 1/2^{n+1}]$, then $x_0 + \lambda h \in I_{n,p}$ and $\lambda h + jh - \frac{1}{2}kh \in E_\varepsilon$ for each $j = 0, 1, \ldots, k$. So performing calculations as in Lemma 2, we have the following:

(1) If $|\lambda| \leq 1$,

$$|\Delta_k(x_0 + \lambda h, h; f)/h^k - f_{(k)}(x_0)| < 2^k \varepsilon.$$

(2) If $|\lambda| > 1$,

$$|\Delta_k(x_0 + \lambda h, h; f)/h^k - f_{(k)}(x_0)| < 2^k |\lambda|^k \varepsilon \leq 2^k (3/2^{n+1}|h|)^k$$
$$\leq 2^k [3k2^{n+1}/2^{n+2} \varepsilon^{1/2k}]^k \varepsilon$$
$$= (3k)^k \varepsilon^{1/2}.$$

If we set $d = (3k)^k$, then regardless of the absolute value of λ we have

$$|\Delta_k(x_0 + \lambda h, h; f)/h^k - f_{(k)}(x_0)| < d\varepsilon^{1/2}.$$

Let

$$W_{h,\varepsilon} = \{x \in I_{n,p} : |\Delta_k(x, h; f)/h^k - f_{(k)}(x_0)| < d\varepsilon^{1/2}\}.$$

Then we have shown so far that for a fixed

$$\tfrac{1}{2}kh \in \left\{\left[-\frac{1}{2^{n+1}}, -\frac{\varepsilon^{1/2k}}{2^{n+1}}\right] \cup \left[\frac{\varepsilon^{1/2k}}{2^{n+1}}, \frac{1}{2^{n+1}}\right]\right\} \cap F_\varepsilon,$$

we have $|W_{h,\varepsilon}| > (1 - (k+1)\varepsilon)/2^{n-1}$. So

$$\left|\left\{\tfrac{1}{2}kh \in I_n : |W_{h,\varepsilon}| > \frac{1 - (k+1)\varepsilon}{2^{n-1}}\right\}\right| > \frac{1 - 2\varepsilon^{1/2k} - \varepsilon}{2^n}.$$

In the beginning of this proof we specified that ε be so small that $1 - (k+1)\varepsilon > \frac{3}{4}$ and $1 - 2\varepsilon^{1/2k} - \varepsilon > \frac{1}{2}$. So we have the following:

$$|\{\tfrac{1}{2}kh \in I_n : |W_{h,\varepsilon}| > \tfrac{1}{2}|I_{n,p}|\}| > \tfrac{1}{2}|I_n|.$$

This together with the definition of $f_n(p/2^n)$ implies that $f_{(k)}(x_0) - d\varepsilon^{1/2} < f_n(p/2^n) < f_{(k)}(x_0) + d\varepsilon^{1/2}$.

In a similar manner we can find an N' such that for $n > N'$ and p such that $p/2^n < x_0 \leq (p+1)/2^n$, we have $f_{(k)}(x_0) - d\varepsilon^{1/2} < f_n((p+1)/2^n) < f_{(k)}(x_0) + d\varepsilon^{1/2}$. We then let $N_0 = \max(N, N')$ and have that, for $n > N_0$, $|f_n(x_0) - f_{(k)}(x_0)| < d\varepsilon^{1/2}$. Hence $f_n(x_0) \to f_{(k)}(x_0)$, and the theorem is proved.

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